Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 20

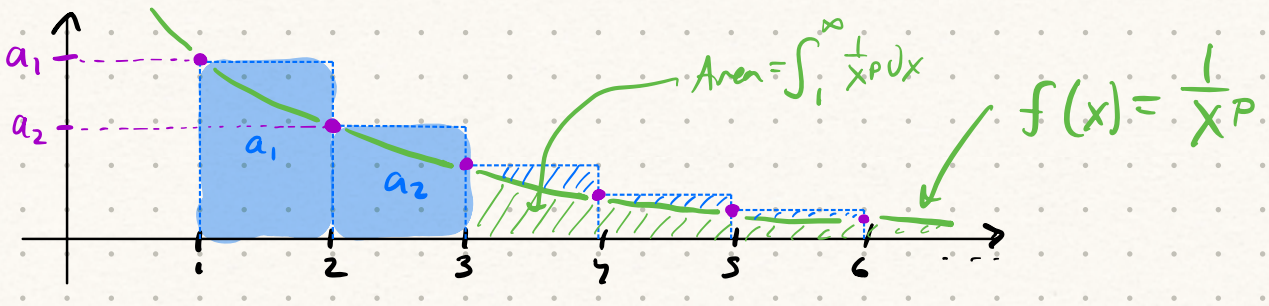
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Textbook Sections: 11.3

Todays Topic: The Integral Tests:

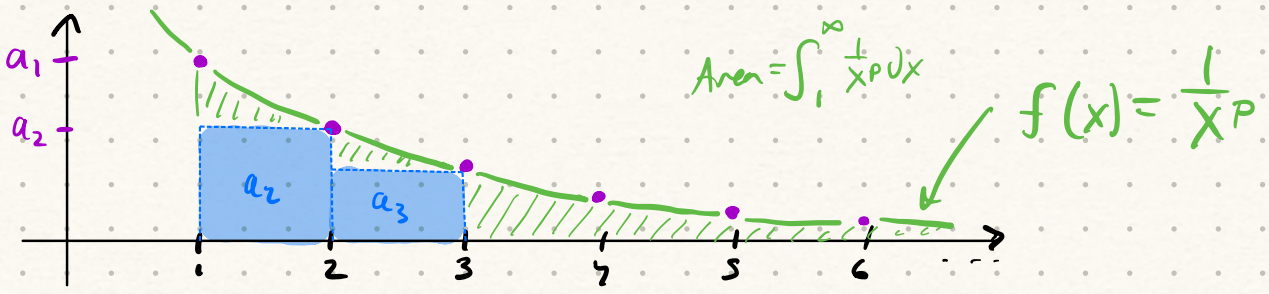
Guiding Question: When does $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge ($p>0$)?

1. Graph the sequence
2. Realize the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ as a sum of areas of rectangles
3. Bound this area with an improper integral which converges/diverges.



$$\sum_{n=1}^{\infty} a_n \geq \int_1^{\infty} \frac{1}{x^p} dx$$

or:



$$\sum_{n=2}^{\infty} a_n \leq \int_1^{\infty} \frac{1}{x^p} dx$$

The point: $\sum_{n=1}^{\infty} a_n \approx \sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $\int_1^{\infty} \frac{1}{x^p} dx$ converges!

We already determined that this integral converges iff $\boxed{p>1}$.

In general, we can turn this argument into another test:

The Integral Test

Suppose that $\sum_{n=1}^{\infty} a_n$ is so that $a_n = f(n)$, for some continuous, positive, decreasing function $f: [1,\infty) \to \mathbb{R}$. Then

$$\sum_{n=1}^{\infty} a_n \text{ is convergent iff } \int_1^{\infty} f(x)\,dx \text{ is convergent.}$$

⚠ Like all of our tests, the hypotheses only need to hold for a "tail" of the sequence $\sum_{n=N}^{\infty} a_n$, i.e. we can infer conv/div. of $\sum_{n=1}^{\infty} a_n$ by showing conv./div. of $\int_N^{\infty} f(x)\,dx$.

Examples: Converge or Diverge?

1) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$.

Soln: Let $f(x) = \frac{\ln x}{x}$ so that $f(n) = a_n$. Notice

- $f(x) > 0$ for $x > 1$
- $f'(x) < 0$ for $x > e$, as $f'(x) = \frac{1 - \ln x}{x^2}$.

So... the integral test applies. We compute

$$\int_1^{\infty} \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty} \tfrac{1}{2}(\ln x)^2 \Big|_1^t = \infty.$$

So: the series must be divergent as well.

2) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

Soln: Let $f(x) = \frac{x^2}{e^{x^3}}$. Then

- $f(x) > 0$ for all $x > 0$
- $f'(x) < 0$ for all $x > (\frac{2}{3})^{1/3}$.

as $f'(x) = \dfrac{2xe^{x^3} - 3x^4 e^{x^3}}{e^{2x^3}} = \dfrac{2x - 3x^4}{e^{x^3}} = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$

if $x > \left(\frac{2}{3}\right)^{1/3}$.

Thus, the integral test applies, and we compute: (note $\left(\frac{2}{3}\right)^{\frac{1}{3}} < 1$)

$$\int_1^\infty \frac{x^2}{e^{x^3}}\,dx = \lim_{t\to\infty} \int_1^t \frac{x^2}{e^{x^3}}\,dx = \lim_{t\to\infty} \int_1^{\sqrt{t}} \frac{1}{3}e^{-u}\,du \quad (u = x^3)$$

$$= \lim_{t\to\infty} \left(-\frac{1}{3}e^{-u}\right)\Big|_1^{\sqrt{t}}$$

$$= \frac{1}{3e}$$

SO... $\sum_{n=1}^\infty \frac{n^2}{e^{n^3}}$ converges as well!

3) $\sum_{n=1}^\infty \frac{\cos^2 n}{1+n^2}$.

⚠ In this case, the integral test does not work with $f(x) = \frac{\cos^2 x}{1+x^2}$, since f is not monotone decreasing!

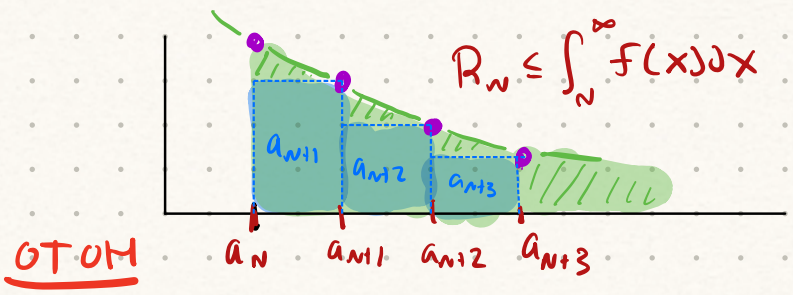
Instead, use a comparison:

$$0 \le a_n = \frac{\cos^2 n}{1+n^2} \le \frac{1}{n^2} = b_n, \text{ and } \sum b_n \text{ converges...}$$

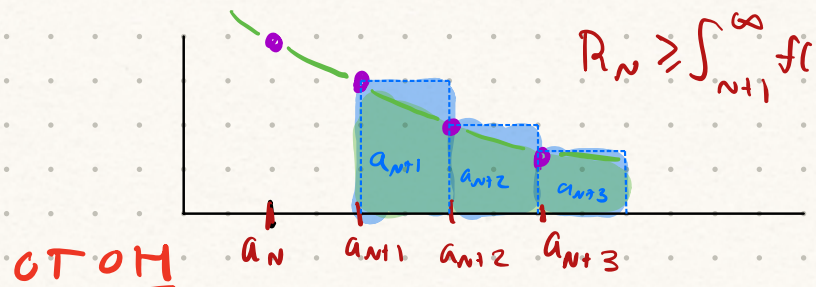
As it stands, the integral test can be helpful in determining when a series converges/diverges, but it doesn't tell use the value of the series when it does converge...

However, the integral test can help us estimate the value of the series, which we shall expect given the earlier pictures we drew!

Suppose $\sum_{n=1}^\infty a_n = S$, and let $R_N = S - S_N = \sum_{n=N+1}^\infty a_n = a_{N+1} + a_{N+2} + \cdots$



GTOH



GTOH

Thus we have the following estimate, which actually approximates S better than the partial sum S_N does:

Integral Test Remainder Estimate: Suppose $f(n)=n \ \forall n \geq N$, and f is cts, decreasing...

$$\int_{N+1}^{\infty} f(x)\,dx \leq R_N \leq \int_N^{\infty} f(x)\,dx$$

or

$$S_N + \int_{N+1}^{\infty} f(x)\,dx \leq S \leq S_N + \int_N^{\infty} f(x)\,dx$$

Example: Let $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Then if $N=5$: $S_5 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25}$

≈ 1.4636

Now (1) $\int_5^{\infty} \frac{1}{x^2}\,dx = \frac{1}{5}$

(2) $\int_6^{\infty} \frac{1}{x^2}\,dx = \frac{1}{6}$

So, we have that

$$1.630 \approx S_5 + \int_6^{\infty} \frac{1}{x^2}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n^2} \leq S_5 + \int_5^{\infty} \frac{1}{x^2}\,dx \approx 1.6636$$

Fact: $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

History Aside: The question of finding the value of this series was posed in 1650 and only solved 84 years later by Euler, when he was... 28. He used infinite series to do this, using some results now known to be rigorous in complex analysis.

Nowadays, many more proofs of this are known, including a proof using Fourier series that takes ~ two lines.